

Measurability in Linear and Non-Linear Quantum Mechanical Systems

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Abstract

The measurability by means of continuous measurements, of an observable $\mathbf{A}(t_0)$, at an instant, and of a time averaged observable, $\bar{\mathbf{A}} = 1/T \int \mathbf{A}(t') dt'$, is examined for linear and in particular for non-linear quantum mechanical systems. We argue that only when the exact (non-perturbative) solution is known, an exact measurement may be possible. A perturbative approach is shown to fail in the non-linear case for measurements with accuracy $\Delta \bar{\mathbf{A}} < \Delta \bar{\mathbf{A}}_{min}(T)$, giving rise to a restriction on the accuracy. Thus, in order to prepare an initial pure state of a non-linear system, by means of a continuous measurement, the exact non-perturbative solution must be known.

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I. INTRODUCTION

The measurement of an observable $\mathbf{A}(t_0)$ is ideally described by an impulsive coupling between the system and the measuring device at $t = t_0$ [1]. In a realistic measurement however, due to the finite duration of the coupling between the systems, the back-reaction of the measuring device modifies the free evolution of the system. Thus, one generally observes a time averaged observable which depends of *both* the system and the measuring device. This raises the following questions:

- Can the *exact* value of $\mathbf{A}(t_0)$ at the instant $t = t_0$, be measured by means of a *continuous* measurement?
- Can the value of a time averaged observable, $\bar{\mathbf{A}} = T^{-1} \int_0^T \mathbf{A}(t') dt'$, be precisely measured?

In this article we shall address these questions in the context of linear and in particular of non-linear quantum mechanical systems. Other aspects of continuous measurements have been discussed for example in Refs. [2–5].

The answer to the first question above has fundamental significance for quantum mechanics. According to quantum mechanics, in order to *prepare* a certain initial pure state $\psi(t = t_0)$, one has to determine the values of a complete set of observables at the *instant* $t = t_0$. But what if only continuous measurements are possible?

It turns out that for linear as well as non-linear systems, as long as the *exact* solution to the Heisenberg equations of motion of the system is known, such a preparation is at least formally possible. To see this, consider a general exact solution for the observable $\mathbf{A}(t)$:

$$\mathbf{A}(t) = \xi(t - t_0, \mathbf{A}_m(t_0)), \tag{1}$$

which is given in terms of initial values of the observables \mathbf{A}_m at time $t = t_0$. By sending $t \rightarrow -t$ and replacing the roles of t and t_0 we can represent the constant of motion, $\mathbf{A}(t = t_0)$, in terms of time dependent operators $\mathbf{A}_m(t)$:

$$\mathbf{A}(t = t_0) = \xi(t_0 - t, \mathbf{A}_m(t)). \quad (2)$$

We can now use the standard continuous measurement described by the interaction

$$H_I = -g(t)\mathbf{Q}\xi(t_0 - t, \mathbf{A}_m(t)), \quad (3)$$

where \mathbf{Q} is conjugate to the “output observable” \mathbf{P} , and $g(t) = g_0$ for $t_0 < t < t_0 + T$ (or more generally, g has a finite support in time, and $\int g dt$ is finite). Since $[\mathbf{A}(t_0), H_I] = 0$, the measurement does not cause an error in $\mathbf{A}(t_0)$, the exact value can be measured. However, this approach requires the knowledge of the exact non-perturbative solution. But what if the latter is unknown? We shall further discuss this issue in Sections 5-7.

The second question raised above becomes particularly acute when asked in the context of quantum field theory. The singular nature of a field at a point requires to consider space-time weighted (smeared) fields as the elementary observables of quantum field theory [6]. Therefore, we are faced with the problem of measuring a space-time averaged observable. The measurability of such objects has been investigated long ago by Bohr and Rosenfeld [7] for the case of a free field, but has not been considered for non-linear field theories [8].

As a toy model we shall consider in this article the problem in the simpler case of non-relativistic quantum theory. For the case of a linear theory, the back-reaction on the system during a measurement of $\bar{\mathbf{A}} = T^{-1} \int \mathbf{A}(t') dt'$ depends only on variables of the measuring device. A compensation term can therefore be devised so as to exactly compensate for the back-reaction, and measure the latter averaged observable to any desired accuracy. But if the system has non-linear equations of motion, the problem of observing a time averaged quantity becomes much more difficult. The error due to the non-linear back-reaction becomes a function of both system and measuring device variables. As already described above, if a non-perturbative solution is known, then formally a precise measurement is possible. Since in general however this is not the case, we shall consider in some details the problem of measuring a time averaged observable by means of a perturbative approach. We shall show that the validity of a perturbative method (in the non-linear coupling constant) is

limited. For measurement with of accuracy better than a certain minimal uncertainty, the perturbation scheme breaks-down.

In the next section we shall present the toy model which will be used in this article to study linear and non-linear cases: a harmonic oscillator with a non-linear potential. In Section 3. we shall consider three different approaches to measure averaged observables in the linear case. We then discuss these methods when the system is non-linear in Section 4. A perturbative approach to the non-linear case is developed in Section 5, and shown to break-down for precise measurements in Section 6. We conclude by a discussion of the main results.

II. NON-LINEAR HARMONIC OSCILLATOR AS A TOY PROBLEM

We shall consider as a toy model for a non-linear system a harmonic oscillator with a potential given by

$$H = \frac{1}{2}(\mathbf{p}^2 + \Omega^2 \mathbf{x}^2) - \frac{\lambda}{n} \mathbf{x}^n, \quad (4)$$

where $n > 2$.

Our first aim will be to describe a measurement of an averaged observable such as

$$\bar{\mathbf{x}} = \frac{1}{T} \int_0^T \mathbf{x}(t') dt'. \quad (5)$$

In the limit of $T \rightarrow 0$ the prescription is known: the appropriate interaction term is in this case

$$H_I = -g(t) \mathbf{Q} \mathbf{x}, \quad (6)$$

where \mathbf{Q} is canonically conjugate to the "output variable" \mathbf{P} , and $g(t) = g_0 \delta(t)$. It is assumed that the effective mass of the device is infinitely large, thus the kinetic part of the measuring device Hamiltonian vanishes and \mathbf{Q} is a constant of motion. The interaction Hamiltonian (6) yields:

$$\delta\mathbf{P} \equiv \mathbf{P}(+\epsilon) - \mathbf{P}(-\epsilon) = g_0\mathbf{x}(0). \quad (7)$$

As a starting point let us modify this interaction by replacing the Dirac-delta function by a smooth function $g(t)$ with a compact support only the time interval $0 < t < T$.

The solution to the equations of motion

$$\dot{\mathbf{x}} = \mathbf{p}$$

$$\dot{\mathbf{p}} = -\Omega^2\mathbf{x} + \lambda\mathbf{x}^{n-1} + g(t)\mathbf{Q}$$

$$\dot{\mathbf{P}} = g(t)\mathbf{x}, \quad (8)$$

can be written in the integral form:

$$\mathbf{x}(t, g) = \mathbf{x}_0(t) + \int_0^t \frac{F(t')}{\Omega} \sin[\Omega(t - t')] dt' \quad (9)$$

$$\mathbf{p}(t, g) = \mathbf{p}_0(t) + \int_0^t F(t') \cos[\Omega(t - t')] dt', \quad (10)$$

where

$$F(t) = \lambda\mathbf{x}^{n-1}(t, g) + g(t)\mathbf{Q}, \quad (11)$$

and $\mathbf{x}_0(t)$, $\mathbf{p}_0(t)$ are free solutions ($\lambda = g = 0$), which coincide with the non-linear solution $\mathbf{x}(t)$ at $t = 0$. The presence of the coupling parameter g in $\mathbf{x}(t, g)$ will be used in the following to denote an explicit dependence of the solution on variables of the measuring device.

The solutions $\mathbf{x}(t, g)$ and $\mathbf{p}(t, g)$ depend on the back-reaction via the term $g(t)\mathbf{Q}$. Contrary to classical mechanics, due to the uncertainty principle, we can not make \mathbf{Q} as small as we wish and still obtain an accurate measurement, i.e. $\Delta\mathbf{P}(0) \rightarrow 0$. Consequently, we finally observe

$$\mathbf{P}(T) - \mathbf{P}(0) = \int_0^T g(t)\mathbf{x}(t, g)dt, \quad (12)$$

which is not the undisturbed ($g=0$) value of $\bar{\mathbf{x}}$. Only in the limiting case $T = 0$ does the error vanish.

In next section we show how the undisturbed $\bar{\mathbf{x}}$ can be observed in the linear case; $\lambda = 0$.

III. THE LINEAR CASE

We shall now proceed to examine the case of a measurement of an averaged observable and show how to eliminate the back reaction in this special case.

For simplicity we shall choose the weight function as $g(t) = g_0$ for $t \in (0, T)$ and zero otherwise. In this case, the solution of the equations of motions when $\lambda = 0$ is

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{x}_0(t) + \frac{g_0(1 - \cos \Omega t)}{\Omega^2} \mathbf{Q} \\ &\equiv \mathbf{x}_0(t) + \alpha(t) \mathbf{Q}.\end{aligned}\tag{13}$$

Thus, the \mathbf{P} coordinate of the measuring device will be shifted by

$$\delta \mathbf{P} = \mathbf{P}(T) - \mathbf{P}(0) = g_0 \int_0^T \mathbf{x}(t') dt' = g_0 \int_0^T \mathbf{x}_0(t') dt' + \mathbf{Q} \frac{g_0^2 T}{\Omega^2} \left(1 - \frac{\sin \Omega T}{\Omega T}\right).\tag{14}$$

The undisturbed average $\bar{\mathbf{x}} = \bar{\mathbf{x}}_0 = T^{-1} \int_0^T \mathbf{x}_0(t') dt'$ is given by

$$\bar{\mathbf{x}}_0 = \frac{\delta \mathbf{P}}{g_0 T} - \frac{g_0}{\Omega^2} \left(1 - \frac{\sin \Omega T}{\Omega T}\right) \mathbf{Q}.\tag{15}$$

Since $\bar{\mathbf{x}}_0$ depends linearly on both \mathbf{P} and \mathbf{Q} , a precise measurement of \mathbf{P} causes a larger uncertainty in the second term above. I.e. it increases the back reaction of the measuring device on the oscillator. Since the uncertainty in $\bar{\mathbf{x}}$ is

$$\Delta \bar{\mathbf{x}}_0 \approx \frac{\Delta \mathbf{P}}{g_0 T} + \frac{g_0}{\omega^2 \Delta \mathbf{P}} \left(1 - \frac{\sin \Omega T}{\Omega T}\right),\tag{16}$$

the minimal uncertainty is

$$\Delta \bar{x}_{min} = \frac{2}{\Omega} \sqrt{1 - \frac{\sin \Omega T}{\Omega T}} \xrightarrow{T \rightarrow 0} \sqrt{\frac{2}{3}} T,\tag{17}$$

which vanishes only in the impulsive limit. We also note that in the limit of $T \rightarrow \infty$ the disturbance does not average out but rather approaches the constant $\frac{2}{\Omega}$.

The direct approach therefore fails to measure $\bar{\mathbf{x}}_0$ precisely. There are however ways to correct or eliminate the error above, and in the following we present three different ways to achieve this goal.

The idea of Bohr and Rosenfeld was to correct the error by adding to the Hamiltonian (6) a new “compensating” term. The error in the shift of \mathbf{P} in eq. (14) appears as linear in \mathbf{Q} , very much like the effect of a spring in the \mathbf{Q} -coordinates. Since the coefficient which multiplies \mathbf{Q} in (14) is known, it is straightforward to compensate for this error by adding to the interaction Hamiltonian (6) a spring term:

$$H_I = -g(t)\mathbf{Q}\mathbf{x} + \frac{1}{2}k\mathbf{Q}^2, \quad (18)$$

where k can be chosen for example as

$$k(t) = \frac{g^2(t)T}{\Omega^2} \left(1 - \frac{\sin \Omega T}{\Omega T}\right). \quad (19)$$

It is straightforward to see that the new spring-term in the equations of motion precisely eliminates the back-reaction of the device on the system. In the limit $T \rightarrow 0$, the compensation is of course not needed since the spring constant vanishes, and we obtain back the ordinary impulsive measurement.

Another approach due to Unruh [9], does not require modification of the interaction (6) but requires instead an additional measuring device. Inspecting eq. (15) we notice that $\bar{\mathbf{x}}_0$ is given on the right hand side in terms of a linear combination of \mathbf{P} and \mathbf{Q} . Therefore, after preparing the measuring device in an initial state with a well defined \mathbf{P} one can measure by means of *another* measuring device the linear combination

$$\frac{1}{g_0 T} \mathbf{P} - \frac{k(T)}{g_0} \mathbf{Q}(T), \quad (20)$$

where k is given in eq. (19). As before this method requires that the back-reaction effect on the system depends only on variables of the measuring device. As we shall see in the next section, both methods apparently fail when non-linearities introduce back-reaction terms which depend also on the system itself.

A third different method to measure $\bar{\mathbf{x}}$ could be the following. By integrating $\mathbf{x}(t)$ we can express $\bar{\mathbf{x}}$ as

$$\frac{1}{T} \int_0^T (\mathbf{x}_0 \cos \Omega t' + \frac{1}{\Omega} \mathbf{p}_0 \sin \Omega t') dt' = \frac{\mathbf{x}_0}{\Omega T} \sin \Omega T + \frac{\mathbf{p}_0}{\Omega^2 T} (1 - \cos \Omega T). \quad (21)$$

Note that \mathbf{x}_0 and \mathbf{p}_0 are constants of motion. By inverting the solutions for the equations of motion for $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{p}_0)$ and $\mathbf{p} = \mathbf{p}(\mathbf{x}_0, \mathbf{p}_0)$ we obtain

$$T\bar{\mathbf{x}}(t) = \frac{2}{\Omega} \sin \Omega T/2 \left[\mathbf{x}(t) \cos \Omega(T/2 - t) + \frac{1}{\Omega} \mathbf{p}(t) \sin \Omega(T/2 - t) \right]. \quad (22)$$

Thus, we have expressed $\bar{\mathbf{x}}$ in terms of a (time dependent) constant of motion. We can now set up a continuous measurement

$$H_I = -g(t)\mathbf{Q}\bar{\mathbf{x}}(t). \quad (23)$$

Clearly the observable $\bar{\mathbf{x}}(t)$ remains a constant of motion and is not disturbed by the interaction. We have thus managed to measure $\bar{\mathbf{x}}$ by a continuous measurement of a constant of motion whose value is identical to $\bar{\mathbf{x}}$ for any t .

IV. NON-LINEAR CASE

The presence of non-linear terms makes the back-reaction dependent on both system and measuring device variables. This makes the problem of constructing the analogous compensation more complicated.

Nevertheless, if the exact solution is known, there is at least a formal way of eliminating completely the back-reaction. To this end we can use the third approach that was described in the last section. Let us denote the exact solutions of the equation of motion by

$$x(t) = \xi(\mathbf{x}_0, \mathbf{p}_0, t), \quad p(t) = \xi'(\mathbf{x}_0, \mathbf{p}_0, t), \quad (24)$$

Then, the average $\bar{\mathbf{x}}$ may be expressed as

$$T\bar{\mathbf{x}} = \int_0^T \xi(\mathbf{x}_0, \mathbf{p}_0, t') dt' = \phi(\mathbf{x}_0, \mathbf{p}_0, T), \quad (25)$$

where the latter observable, ϕ , is a constant of motion. We can now substitute

$$\mathbf{x}_0 = \xi(\mathbf{p}(t), \mathbf{x}(t), -t), \quad \mathbf{p}_0 = \xi'(\mathbf{p}(t), \mathbf{x}(t), -t), \quad (26)$$

into eq. (25) and express ϕ in terms of the time dependent dynamical variables:

$$\bar{\mathbf{x}} = \phi(\xi(\mathbf{p}(t), \mathbf{x}(t), -t), \xi'(\mathbf{p}(t), \mathbf{x}(t), -t), T) = \phi(\mathbf{x}(t), \mathbf{p}(t), t, T). \quad (27)$$

Since $\phi(\mathbf{x}(t), \mathbf{p}(t), t, T)$ is a constant of motion we can measure it by means of the ordinary interaction (6).

The other two methods of Section 3. are much harder to use. Assume again the exact solution is given, and let us express it as

$$\mathbf{x}(t, g) = \xi(t, \lambda, g = 0) + \Delta\xi(t, \lambda, g). \quad (28)$$

Here, $\Delta\xi(t, \lambda, g) = \xi(t, \lambda, g) - \xi(t, \lambda, g = 0) = x(t, g) - x(t, g = 0)$ is the non-linear error due to the measurement. Thus, using the naive coupling (6) we obtain:

$$\frac{\delta\mathbf{P}}{g_0T} = \bar{\mathbf{x}} + \frac{1}{g_0T} \int_0^T \Delta\xi(t, \lambda, g) dt'. \quad (29)$$

To proceed, we would like to construct a compensation to the last term.¹ Thus, $\Delta\xi$ must be expressed in terms of the dynamical variables $\mathbf{x}(t, g)$, $\mathbf{p}(t, g)$ and \mathbf{Q} . However, $\Delta\xi$ depends on $\mathbf{x}(t, g = 0)$ and we first need to find the non-linear relation between the disturbed non-linear $\mathbf{x}(t, g)$ and the undisturbed $\mathbf{x}(t, g = 0)$ solutions:

$$\mathbf{x}(t, g) = f[\mathbf{x}(t, 0), \mathbf{p}(t, 0), \mathbf{Q}], \quad (30)$$

etc. The latter equation corresponds to a “canonical” transformation between two non-linear solutions of two different systems. It is not clear if such a well defined relation indeed exist. Assuming however that transformation above does exist, we may now attempt, as before, to add to the Hamiltonian the compensation term $\int[k\mathbf{Q} + \Delta\xi(\lambda, g)]dq$. However, the new equations of motions will give rise to another non-linear error $\Delta\xi' \neq \Delta\xi$. Therefore, a self consistent scheme for constructing a compensation must be found, which takes deals with the non-perturbative effects of the compensation on the system.

¹ At this point we note that in principle we could have arranged that during the measurement a compensation term $-g(t)\lambda\mathbf{x}^n/n$ eliminates the non-linearity. We could then use the previous method for observing the time averaged observable corresponding to a linear dynamics.

The difficulties in finding an exact non-perturbative compensation scheme, leads us to examine a more modest approach. It can be hoped that at least for small averaging time T , the effect of the non-linearities is small. Therefore, in the next section, we shall examine a perturbative approach.

V. A PERTURBATIVE APPROACH

For a small non-linear constant, λ , we shall attempt a perturbative approach. As a starting point we shall assume that the measurement can be expressed by:

$$H_I = -g(t)\mathbf{Q}\mathbf{x} + H_C, \quad (31)$$

i.e., as a sum of the naive measurement and a compensation term. Clearly in the limit of $T \rightarrow 0$, H_I should reduce to an ordinary impulsive measurement, i.e., $H_C(T = 0) = 0$. Our aim will be to construct a self-consistent procedure to compute the compensation term H_C to any order in λ :

$$H_C = \Delta H^{(0)} + \sum_k \lambda^k \Delta H^{(k)}. \quad (32)$$

The terms $\Delta H^{(k)}$ are the analogous compensations to the back-reaction up to the k 'th order in λ .

To this end, let us expand the solution for $\mathbf{x}(t)$ as

$$\mathbf{x}(t, g) = \mathbf{x}^{(0)} + \lambda \mathbf{x}^{(1)}(t) + \lambda^2 \mathbf{x}^{(2)}(t) + \dots \quad (33)$$

and at each order separate the undisturbed solution, denoted as $\mathbf{x}_0^{(k)}$, from the non-linear error, denoted as $\Delta \mathbf{x}^{(k)}$:

$$\mathbf{x}^{(k)}(g) = \mathbf{x}_0^{(k)}(t) + \Delta \mathbf{x}^{(k)}(t, g), \quad (34)$$

i.e. for $g = 0$ we have $\Delta \mathbf{x}^{(k)}(t, g = 0) = 0$ and $\mathbf{x}^{(k)}(g = 0) = \mathbf{x}_0^{(k)}(t)$ is the solution without the coupling to a measuring device.

Likewise, the shift of the output register , $\delta\mathbf{P} = \mathbf{P}(T) - \mathbf{P}(0)$, will be expanded in powers of λ , and at each order we shall evaluate the terms of the decomposition:

$$\delta\mathbf{P}^{(k)} = \delta\mathbf{P}_0^{(k)} + \Delta\mathbf{P}^{(k)}. \quad (35)$$

Here, $\Delta\mathbf{P}^{(k)}$ is the error due to the back reaction to the k 'th order. The knowledge of $\Delta\mathbf{P}^{(k)}$ will allow us to construct the appropriate compensation $\Delta H^{(k)}$, up to the same order k .

To proceed we shall now simplify the problem, and let $n = 3$ in eq. (4). It can be shown however that the following will be valid for any general non-linear potential.

To evaluate $\Delta\mathbf{P}^{(k)}$ we first use eqs. (9) to obtain an integral solution to any order:

$$\mathbf{x}^{(0)}(t, g) = \mathbf{x}_0(t) + \mathbf{Q} \int_0^t g(t') D(t - t') dt', \quad (36)$$

$$\mathbf{x}^{(1)}(t, g) = \int_0^t \mathbf{x}^{(0)}(t') \mathbf{x}^{(0)}(t') D(t - t') dt', \quad (37)$$

$$\mathbf{x}^{(2)}(t, g) = 2 \int_0^t \mathbf{x}^{(0)}(t') \mathbf{x}^{(1)}(t') D(t - t') dt', \quad (38)$$

etc., where \mathbf{x}_0 is the free solution ($g = \lambda = 0$), and

$$D(t) = \frac{\sin \Omega t}{\Omega}. \quad (39)$$

The result for the zeroth order was already obtained above, where we found

$$\Delta H_0 = \frac{1}{2} k \mathbf{Q}^2, \quad (40)$$

where k is the c-number given by eq. (19). Here, and in the following we shall ignore any potential problems with ordering of non-commuting operators.

To the first order in λ we obtain

$$\delta\mathbf{P}^{(1)} = g_0 \int_0^T dt \int_0^t [\mathbf{x}^{(0)}]^2 dt'. \quad (41)$$

Thus,

$$\Delta\mathbf{P}^{(1)} = g_0 \int_0^T dt \int_0^t \left[2\alpha(t') \mathbf{Q} \mathbf{x}_0(t') + \alpha^2(t') \mathbf{Q}^2 \right] dt'$$

$$\begin{aligned}
&= g_0 \int_0^\infty \left[2\alpha(t') \mathbf{Q} \mathbf{x}_0(t') + \alpha^2(t') \mathbf{Q}^2 \right] dt' \int_0^T \theta(t-t') dt \\
&= g_0 \int_0^\infty \left((T-t')\theta(T-t') - (-t')\theta(-t') \right) \left[2\alpha(t') \mathbf{Q} \mathbf{x}_0(t') + \alpha^2(t') \mathbf{Q}^2 \right] dt' \\
&= g_0 \int_0^T (T-t') \left[2\alpha(t') \mathbf{Q} (\mathbf{x}^{(0)}(t') - \mathbf{Q} \alpha(t')) + \alpha^2(t') \mathbf{Q}^2 \right] dt' \\
&= g_0 \int_0^T (T-t') \left[2\alpha(t') \mathbf{Q} \mathbf{x}^{(0)}(t') dt' - \alpha^2(t') (T) \mathbf{Q}^2 \right] dt'. \tag{42}
\end{aligned}$$

where $\alpha(t)$ is defined in eq. (13). As could have been anticipated, the new feature of the error terms up to the first order in λ , is their depends on the variable \mathbf{x} of the system. The last term does not depend on system variables and therefore may be trivially compensated as a spring-like compensation. (Alternatively, as discussed in Section 3., we may use a second measuring device to measure the combination $\frac{1}{g_0 T} \mathbf{P} - k \mathbf{Q} - k'(T) \mathbf{Q}^2$, where k' can be found by integrating the last term in eq. (42).)

Equation (42) suggests adding as a compensation the term

$$\Delta H^{(1)} = g^2(t)(T-t) \left[\alpha(t) \mathbf{Q}^2 \mathbf{x}(t) - \frac{1}{3} \alpha^2(t) \mathbf{Q}^3 \right]. \tag{43}$$

With the compensation $\Delta H^{(0)} + \lambda \Delta H^{(1)}$ added to the Hamiltonian, the eqs. of motion become

$$\dot{\mathbf{P}} = g(t) \mathbf{x} - k(t) \mathbf{Q} + \lambda g^2(t)(T-t) \left[2\alpha(t) \mathbf{Q} \mathbf{x}(t) - \alpha^2(t) \mathbf{Q}^2 \right], \tag{44}$$

$$\frac{d^2 \mathbf{x}}{dt^2} + \Omega^2 \mathbf{x} = \lambda \mathbf{x}^2 + g(t) \mathbf{Q} + \lambda g^2(t)(T-t) \alpha(t) \mathbf{Q}^2. \tag{45}$$

The new term on the right hand side of (45) modifies the solution for $x(t)$ to

$$\mathbf{x}(t) = \mathbf{x}_0(t) + \alpha(t) \mathbf{Q} + \lambda \beta(t) \mathbf{Q}^2 + \lambda \int_0^T \mathbf{x}^2(t') dt'. \tag{46}$$

The first order is modified by the $\beta(t) \mathbf{Q}^2$ which can trivially be compensated by adding also the term $\frac{1}{3} \lambda g(t) \beta(t) \mathbf{Q}^3$. This completes the compensation to the first order. It is important to note that the a first order compensation can not modify lower order compensations but only higher orders. This allows us to go to higher orders in λ without effecting the corrections found at lower orders.

Let us see how one can proceed to higher orders. To the second order we obtain:

$$\frac{d\Delta\mathbf{P}^{(2)}}{dt} = g(t)\mathbf{Q}\Delta\mathbf{x}^{(2)} + 2g^2(t)(T-t)\alpha(t)\mathbf{Q}\Delta\mathbf{x}^{(1)}(t), \quad (47)$$

where

$$\Delta\mathbf{x}^{(1)}(t) = \beta(t)\mathbf{Q}^2 + \int_0^t \Delta[\mathbf{x}^{(0)}(t')]^2 dt', \quad (48)$$

and

$$\Delta\mathbf{x}^{(2)}(t) = 2 \int_0^t \Delta[\mathbf{x}^{(0)}(t')\mathbf{x}^{(1)}(t')] dt' \quad (49)$$

where the notation $\Delta[\dots]$, on the right hand side in the eqs. above, indicates that only \mathbf{Q} -dependent terms of $[\dots]$ are included.

The computation can be carried out, as before. The only technical subtlety is that now $\Delta\mathbf{P}^{(2)}$ contain also triple integrals over time like: $\int dt \int x_0(t') dt' \int p_f(t'') dt''$. To proceed one has to reduce this integral to a single integration over time, or eliminate the integrals completely. Otherwise, the compensations would have to be non-local in time. In our case of a first quantized theory, we can always use the property of linear operators (but not field operators): $\mathbf{p}_0(t)$ (or $\mathbf{x}_0(t)$) can always be expressed as a linear combination of $\mathbf{x}_0(t=0)$ and $\mathbf{p}_0(t=0)$. Therefore, we can integrate over free operators and reduce: $\int_0^{t'} \mathbf{p}_0(t'') dt'' = c_1(t')\mathbf{x}_0(t') + c_2(t')\mathbf{p}_0(t')$. The integral above can be hence reduced to a single integral of the form $\int (c'_1(t)\mathbf{x}_0^2 + c'_2(t)\mathbf{x}_0\mathbf{p}_0) dt$. Finally we find that the compensation to the second order has the following form:

$$\Delta H^{(2)} = \mathbf{Q}^2(\gamma_1\{\mathbf{x}, \mathbf{p}\} + \gamma_2\mathbf{x}^2) + \mathbf{Q}^3(\gamma_3\mathbf{x} + \gamma_4\mathbf{p}) + \gamma_5\mathbf{Q}^4. \quad (50)$$

where $\gamma_i(T)$ are generally time dependent c-numbers. In principle it seems that this procedure can be carried out to any order in λ .

VI. BREAK-DOWN OF THE PERTURBATIVE APPROACH

Although by using a perturbative approach we can formally evaluate self-consistently compensation terms to any order in the nonlinear coupling constant, this approach still does

not allow precise measurements. Unlike classical measurements, in the limit of a precise measurement, the variable \mathbf{Q} , which is conjugate to the “output register” \mathbf{P} , becomes completely uncertain. Since the expansion for the compensation H_C generally depends on \mathbf{Q} , for a given T , the validity of the perturbative expansion must break-down beyond a certain precision $\Delta\mathbf{x}_{min}$.

To show this consider the first order compensation term. From eq. (43) we have $\Delta H^{(1)} \approx T\alpha(T)\mathbf{Q}^2\mathbf{x}$. A necessary (but not sufficient) condition for a well behaved expansion is therefore:

$$\lambda T\alpha(T)\mathbf{Q}^2\mathbf{x} \sim \lambda T^3\mathbf{Q}^2\mathbf{x} \ll 1. \quad (51)$$

Substituting a rough approximation $\mathbf{Q}^2 \sim (\Delta\mathbf{Q})^2$ and $\mathbf{x} \sim \Delta x$ and using $\Delta\mathbf{P} \sim \Delta\mathbf{x}/g_0T$ we obtain the condition

$$\Delta\mathbf{x} > \Delta\mathbf{x}_{min} \approx \lambda g_0^2 T^5. \quad (52)$$

Only by letting $T \rightarrow 0$ or $\lambda \rightarrow 0$ we regain $\Delta\mathbf{x}_{min} \rightarrow 0$.

VII. DISCUSSION

In this article we have examined the measurability of observables by means of a continuous measurement for linear and non-linear theories. We have seen that as long as the exact non-perturbative solution is known it is possible, at least in principle, to devise an exact measurement. The formal answer to the two questions asked in the introduction is therefore positive.

Nevertheless, the surprise is that without the knowledge of the exact non-perturbative solution, a perturbative approach can not be used to obtain arbitrary accurate measurements. For any non-vanishing value of the non-linear coupling constant, and a given interaction time T , the perturbation breaks-down for any measurement with accuracy better than $\Delta\bar{\mathbf{A}} < \Delta\bar{\mathbf{A}}_{min}(T, \lambda)$. The origin of this failure is that the perturbation to any order, necessarily involves terms which depend on \mathbf{Q} , the conjugate to the output variable

P. Therefore for sufficiently precise measurements, $\Delta\mathbf{P}$, can be arbitrarily small, and $\Delta\mathbf{Q}$ becomes arbitrarily large. At a certain accuracy the perturbation is bound to break-down.

This conclusions clearly applies to the question of measurability of an observable at an instant by means of a continuous measurement. If the exact solution is not known, we shall not be able to compensate perturbatively for the error. Thus, under the restriction of finite time measurements, a preparation of a certain *pure state* ψ of a non-linear system, requires the knowledge of the exact non-perturbative solution of the equations of motion.

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